TWMS J. Pure Appl. Math., V.2, N.1, 2011, pp.74-96

INTRODUCTION TO A FEW METRIC ASPECTS OF FOLIATION THEORY

RÉMI LANGEVIN[1]

ABSTRACT. Foliations can be studied from a dynamical viewpoint, folowing holonomy maps. Here we focus on the geometry of the leaves of codimension one foliations of surfaces of 3-manifolds of constant curvature. The fact that the ambient space is of constant curvature allows us to play the integral geometry games, that is slice with lines, planes etc. We can also consider globally contact points with families of lines, planes etc. That way we obtain theorems about curvature functions defined by the leaves of our foliations.

Keywords: foliation, Reeb foliation, curvature, Gauss curvature, Gauss map, polar curves, isolated singularity.

AMS Subject Classification: 53C12, 57R30, 53C65.

1. Introduction

The name "feuilletage" in French recalls leaves piled up on the ground in autumn. The idea is to fill a manifold with submanifolds which locally look like piled plates (foliation people call them *plaques*).

The notion was invented in the 1940's by by Ch. Ehreshman (see [8] and [9]) (according Reeb), and soon basic theorems were proved by Ehreshman and Reeb (see [8], Ehresmann [9]) and G. Reeb (see [22]); the notion was also known by Chevalley [5] p. 68 in the context of Lie groups). After Poincaré, Painlevé and Dulac, an earlier work of Kaplan ([11]) already considers families of curves filling the plane out of the context of solutions of differential equations.

Our goal will be to study foliations of codimension 1 of the Euclidean plane, the Euclidean 3-dimensional space or open domains contained in these Euclidean spaces, flat tori of dimension 2 or 3 and spheres of dimension 2 or 3 endowed with the standard metric (that is the metric of constant curvature 1). Hyperbolic spaces will be just mentioned.

We will be interested mainly in the geometry of the leaves. For an introduction to the transverse viewpoint see for example [6].

2. Codimension 1 foliations of the plane or of the torus

2.1. **Before foliations.** Let us first give a few ways of filling regions of the plane or the entire plane by curves.

[1] Institut de Mathématiques de Bourgogne, UMR CNRS 5584, Université de Bourgogne, Dijon, France
e-mail: langevin@u-bourgogne.fr
Manuscript received 10 October 2009.

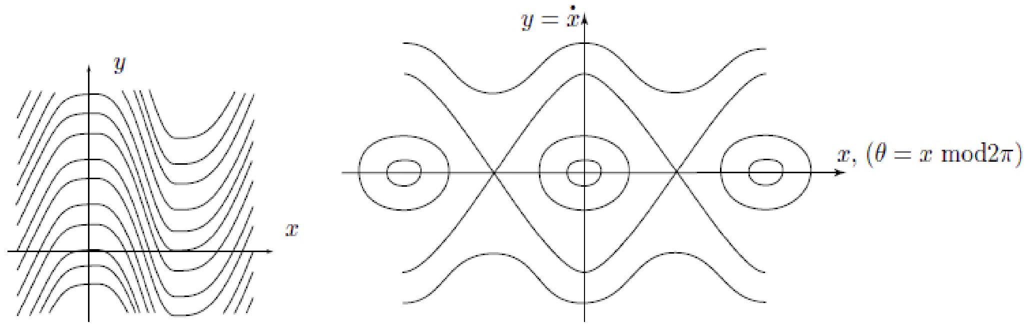


Figure 1. Graph of solutions of $y' = f(x)$ and trajectories of a pendulum in the phase space

Levels of a smooth function $F : \mathbb{R}^2 \to \mathbb{R}$. Let $Crit$ be the set of critical points of F. Then the levels of F foliate $\mathbb{R}^2 \setminus Crit$.

- Graphs of solutions of differential equations

A very simple example is the equation $y' = f(x)$ (a particular case of $y' = f(x,t)$), then the solutions are $\{y = F(x) + c\}$ where $F'(x) = f(x)$.

- Orbits of a smooth vector field

$$\begin{aligned} \dot{x} &= f_1(x,y) \\ \dot{y} &= f_2(x,y). \end{aligned}$$

Many examples come from physics, for example, trajectories of the movement of a pendulum in $\mathbb{R}^2$, a covering of the phase space $S^1 \times \mathbb{R}$. These trajectories define a foliation of $\mathbb{R}^2$ with isolated punctures. These special points, corresponding to equilibria of the movement of the pendulum, are also called *singular points* of the foliation.

Recall the existence and unicity of solution implies that orbits of a smooth vector field with no zero in the domain $W \subset \mathbb{R}^2$ form a family of disjoint curve filling the domain W. This proves that a non-zero vector field X defined on an open subset $W \subset \mathbb{R}^2$ provides a foliation of W. Moreover, both vector fields X and $Y = f \cdot X,\ f \neq 0$, provide the same foliation.

Notice that if we change the vector field multiplying it by a strictly positive function, the integral curves do not change, only their parametrization does.

- Instead of a vector field we may start from a line field: to each point m of the ambient space $W \subset \mathbb{R}^2$ corresponds a direction $\ell(m)$ of $T_mW \simeq \mathbb{R}^2$.

2.2. **General definitions.** In order to give a formal definition of a foliation of a manifold, we need to start, as for the definition of a manifold, from a covering $\mathcal{V} = \{V_i\}$ of a Hausdorff topological space M, and homeomorphisms

$$\phi_i : V_i \to U_i;\ U_i \simeq]0,1[^n \subset \mathbb{R}^n.$$

The previous line says that the model of an n-dimensional manifold is the open cube $]0,1[^n$. The definition of a manifolds uses gluing maps $h_{ij} : U_i \to U_j$ defined each time $V_i \cap V_j \neq \emptyset$.

For those who love indices, we can define chart maps $\phi_i : U_i \to V_i$ and $\phi_j : U_j \to V_j$ and $h_{ij} = \phi_j \circ \phi_i^{-1}$ where it makes sense, that is from $(\phi_i)(V_i \cap V_j)$ to $(\phi_j)(V_i \cap V_j)$.

In order to get a differentiable, smooth or analytic manifold, it is enough to impose that the maps h_{ij} are differentiable, smooth or analytic respectively . In order to define a foliation of dimension p of a differentiable manifold M^n of dimension n (we will not here consider topological foliations), we need to split the model open cube $]0,1[^n$ into $]0,1[^n =]0,1[^p \times]0,1[^q$, $p + q = n$, and impose that the diffeomorphisms h_{ij} send horizontal levels $(]0,1[^p \times y,\ (y \in]0,1[^q))$ of U_i to horizontal levels of U_j. We can give a formula, using $x_1 \in]0,1[^p$ and $y_1 \in]0,1[^q$ in the source U_i and $x_2 \in]0,1[^p$ and $y_2 \in]0,1[^q$ in the target U_j of the map h_{ij}

$$x_2 = h^1_{ij}(x_1, y_1),\ y_2 = h^2_{ij}(y_1).$$

Let us insist: saying that y_2 does not depend on x_1 but only on y_1 means that the map h_{ij} send an horizontal level $]0,1[^p \times y_1$ of U_1 to the horizontal level $]0,1[^p \times y_2$, $y_2 = h^2_{ij}(y_1)$ of U_2.

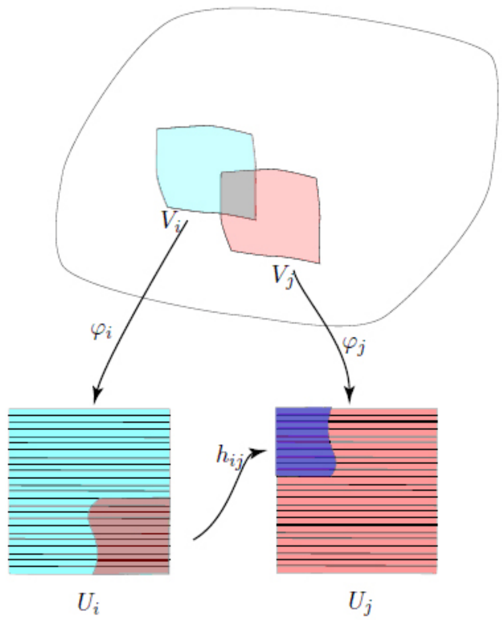


Figure 2. Charts defining a foliation ($n = 2$, $p = q = 1$)

Another way to think of charts is to equip the open sets V_i with submersions p_i to segments (the vertical segment in the model). We leaves the compatibility conditions to the reader. A *plaque* of a foliation associated to a chart (V_i, U_i, ϕ_i) is an image $\phi_i(]0,1[^p \times y)$ of an horizontal level of the model $U_i =]0,1[^p \times]0,1[^q$.

Example 2.1. Parallel lines in $\mathbb{R}^2$ and the quotient foliation in $\mathrm{T}^2 = \mathbb{R}^2/(1,0)\cdot\mathbb{Z} \oplus (0,1)\cdot\mathbb{Z}$. Charts can be obtained from rectangles choosing the sides small enough.

Example 2.2. Let A be the annulus of the Euclidean plane of boundary the circles of center the origin, and radii respectively 1 and 2. The leaves of the foliation spiral towards the two boundary components, but an orientation of a leaf provide orientations of the boundary circles which "turn" in different directions (see Figure 3).

We will call such a foliated annulus a *Poincaré component we keep the name "Reeb component" for the foliation of the solid torus of Fig.18.*

A *leaf* of a foliation is obtained starting from a plaque, gluing to it the adjacent plaques etc... More formally, one can define a topology on the ambient manifold with a basis of open sets the open sets of the plaques of the foliation. A leaf is now a connected component for this topology. It is convenient to accept a few exceptional points that we call *singular points.*

Figure 3. A foliated annulus: a *Poincaré component*

2.3. **Foliations with isolated singularities.**

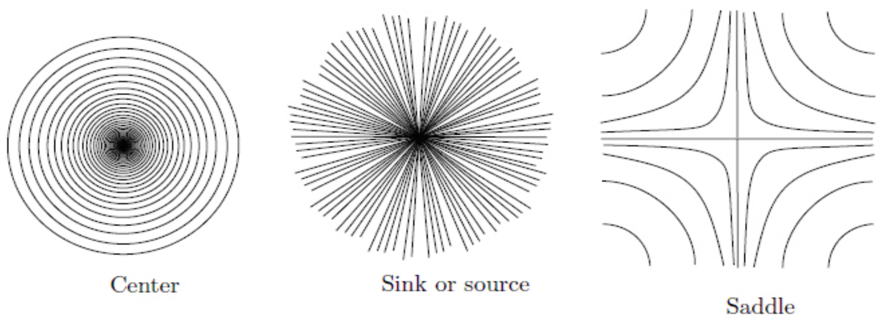


Figure 4. Center, sink or source and saddle

We say that we have a foliation of manifold with singularities if we have a foliation of a manifold, deprived of a lower dimensional set of points like a finite union of smooth hypersurfaces maybe with boundary. When the ambient manifold is of dimension 2, we suppose that the singular points are isolated.

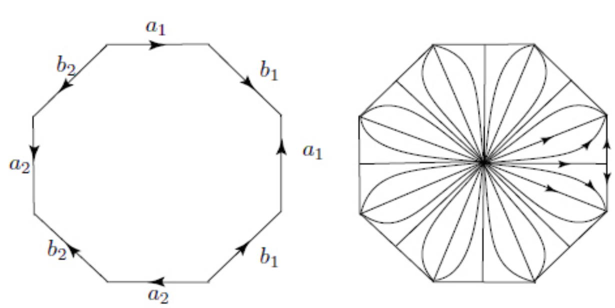


Figure 5. A foliation of a surface of genus 2 with 6 singularities

Using vector fields in the plane, one can define some particular singularities.
- $X_1(x, y) = (-x, -y)$ has a sink at the origin.
- $X_2(x, y) = (x, y)$ has a source at the origin.
- $X_3(x, y) = (x, -y)$ has a saddle at the origin.
- $X_4(x, y) = (-y, x)$ has a center at the origin.

Example 2.3. *A surface of genus* 2, Σ_2*, can be obtained from a regular octagon identifying the sides as indicated on Figure 5. We can draw the orbits of a vector field on the octagon which has* *a source at the center of symmetry of the octagon, two saddle-like sectors touching the middle of each side, and a sink-like sector at each edge. After the identifications we get on Σ_2 one source, one sink and four saddles.*

2.3.1. *Non-orientable singularities.*

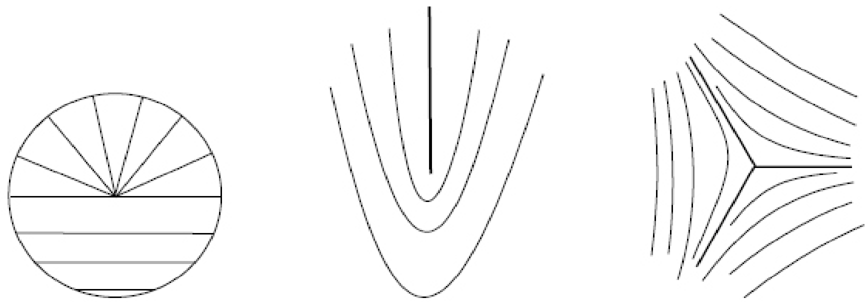


Figure 6. Non-orientable singularities: sunset, thorn and a 3-prongs saddle

A foliation can always be oriented in a neighbourhood of a regular point. This is not the case in the neigbourhood of a singular point as one can see on Figure 2.3.1. The examples comme from a *line-field*, that is the assignment of a line at each point of the foliated domain. Following a small circle around the singular point, we see that it is impossible on our examples to define an orientation of the lines on a neighbourhood of the singular point.

2.4. **Exchange theorem.** We consider now a smooth foliation of a domain $U \subset \mathbb{R}^2$. We will study globally the "amount of curvature" allowed by the shape of the domain when the boundary of the domain is supposed to be a union of leaves.

Through a point $x \in U$ goes a leaf L. If we orient the leaf L, it has a curvature $k(x)$ at the point x. The *total curvature* of the foliation $\mathcal{F}$ is the integral $\int_W |k|dv$. Notice that we need not to orient simultaneously in a compatible way all the leaves, as we consider only $|k|$. Notice also that we do not integrate the curvature leaf by leaf, we consider the integral of the function $|k|$ on the whole domain U, endowed with the (2-dimensional) Lebesgue measure. We will denote by $T_m\mathcal{F}$ the line tangent at m to the leaf L_m of $\mathcal{F}$.

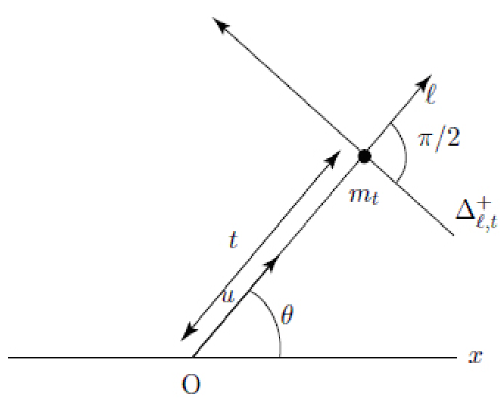


Figure 7. Oriented directions

The space of affine lines of the Euclidean plane is a 2-dimensional manifold endowed with a natural measure. An exhaustive reference is Santaló's book, chapter 3 [23]. Given an origin of the Euclidean plane, an oriented line is defined by an angle $\theta \in S^1$ and the coordinate of a point in the oriented line through the origin making the angle θ with the x-axis (see figure 7). The density we will use on the set of oriented affine line $\mathcal{A}^+(2,1)$ is then $d\theta \otimes dt$. Notice that this measure is invariant by the action of the group of affine isometries. The measure on the set $\mathcal{A}(2,1)$ of non-oriented lines is essentially the same. Usually one takes half of the image measure of $d\theta \otimes dt$ by the 2 to 1 map from $\mathcal{A}^+(2,1)$ to $\mathcal{A}(2,1)$ obtained forgetting the orientation of the line.

An "exchange theorem" relates the number $|\mu|(\mathcal{F}, H)$ of contact of a foliation with affine lines $H \in \mathcal{A}(2,1)$ and the total curvature of $\mathcal{F}$.

Theorem 2.1. *(Foliated exchange theorem).*

$$\int_W |k| dv = \int_{\mathcal{A}(2,1)} |\mu|(\mathcal{F}, H).$$

To prove this theorem, we will define the *polar curves* of the foliation and a foliated Gauss map.

Polar curves. The critical points of the orthogonal projection of a leaf L of $\mathcal{F}$ on a line ℓ are in general isolated on the leaf L.

Proposition 2.1. *The* polar set*, closure of the union of these critical points for all the leaves of the foliation*

$$\Gamma(\mathcal{F}, \ell) = \overline{\bigcup_L crit(p_\ell|_L)}$$

is generically almost everywhere a smooth curve (it may have singular points). We will call it then a polar curve *of the foliation $\mathcal{F}$.*

Remark 2.1. *Notice that we can also define the polar set $\Gamma(\mathcal{F}, \ell)$ as the closure of the set where the foliation $\mathcal{F}$ is tangent to the foliation of the plane by the affine lines orthogonal to ℓ, that is the set of points m where $T_m\mathcal{F}$ is a line orthogonal to ℓ, see Figure 2.4.*

Proof. Using the theorem of Sard-Brown we see that the map $\gamma : W \to \mathbb{P}^1$ which to a point $m \in W$ associates the tangent direction at m to the leaf through m has, when the foliation is smooth enough, a set of critical values of measure zero. Therefore, for a regular value ℓ of γ, Γ_ℓ is a smooth curve.

We include a singularity of the foliation in a curve Γ_ℓ if it belongs to the closure of the set of points $m \in W$ such that ℓ is the tangent direction at m to the leaf through m. When the dimension of the ambient space is 2, we will suppose that the singular points of the foliation are isolated. □

Notice that on the set C_γ of critical points of $\gamma : W \to \mathbb{P}^1$, one has $k = 0$. Therefore γ^{-1}(critical values of γ) is the union of a set of measure zero and a set where $k = 0$.

Remark 2.2. *When $\Gamma(\mathcal{F}, T_m\mathcal{F}^\perp)$ is tangent to $T_m\mathcal{F}$ the Gauss curvature of the leaf L_m is zero, as, in that case, the differential of the Gauss map of the leaf L_m restricted to $T_m\Gamma(\mathcal{F}, T_m\mathcal{F}^\perp)$ is zero.*

To prove the foliated exchange theorem we need also to introduce a foliated Gauss map with values in $\mathcal{A}(2,1)$.

Definition 2.1.

$$\gamma_{\mathcal{F}}(m) = \textit{the affine line tangent at } m \textit{ to } \mathcal{F}.$$

Proof of the theorem 2.1: The previous remark implies that, excluding points where $k = 0$, which do not contribute to the integral $\int_W |k(m)|dm$, we exclude the points where Γ_ℓ is tangent to L at m.

To compute the Jacobian determinant of the foliated Gauss map $\gamma_{\mathcal{F}}$ at a point $m \in (W \setminus \{m|k(m) = 0\}$ we will use, as then $\Gamma(\mathcal{F}, T_m\mathcal{F}^\perp)$ is transverse to $T_m\mathcal{F}$ in the domain, the frame u_1, u_2, where u_1 is an orthogonal basis of $T_m\mathcal{F}$, and u_2 is the unit vector tangent at m to $\Gamma(\mathcal{F}, T_m\mathcal{F}^\perp)$. In $\mathcal{A}(2,1)$ we use at $\gamma_{\mathcal{F}}(m)$ the frame v_1, v_2, v_3, where v_1 form an orthogonal basis of the horizontal space at $\gamma_{\mathcal{F}}(m)$ of the Riemannian fiber bundle $\mathcal{A}(2,1) \to \mathbb{P}^1$ (the projection map associates to an affine line its parallel through the origin), and where v_2 is a unit vector tangent to the fiber of $\mathcal{A}(2,1) \to \mathbb{P}^1$. In these bases, the matrix of $d\gamma_{\mathcal{F}}$ is

$$\begin{pmatrix} d\gamma_{\mathcal{F}}|_{L_m} & 0 \\ * & |\cos\phi| \end{pmatrix},$$

where ϕ is the angle between $T_m\Gamma_{\mathcal{F}}$ and $T_m\mathcal{F}^\perp$.

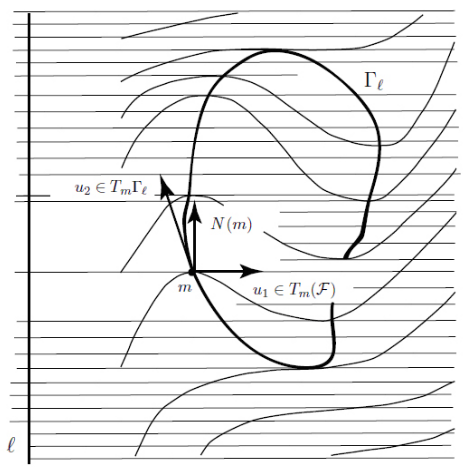


Figure 8. Computation of the Jacobian determinant of $\gamma_{\mathcal{F}}$ in dimension 2

As the volume of the parallelogram determined by the frame u_1, u_2 is also $|\cos\phi|$, and as the map $d\gamma_{\mathcal{F}}|_{L_m}$ is just the Gauss-Kronecker map of the leaf L_m, the Jacobian determinant we are looking for is $|k|$. Notice that the previous computation implies that critical points of $\gamma_{\mathcal{F}}$ are contained in the set $k = 0$.

We can neglect the points of C_γ, union of a measure zero set and a set where $k = 0$, and the points of $\gamma_{\mathcal{F}}^{-1}$(critical values of $\gamma_{\mathcal{F}}$), which is also a union of a measure zero set (regular points of $\gamma_{\mathcal{F}}$ of image a critical value of $\gamma_{\mathcal{F}}$), and a set (containing C_γ) where $k = 0$. Avoiding all the points where $k = 0$, we can use a frame split between $T_m\mathcal{F}$ and $T_m\mathcal{F}^\perp$. We can now divide the complement of this set of "bad" points into an enumerable union of open sets U_i where γ_{fol} is a diffeomorphism, throwing away if necessary another measure zero subset of W. The exchange theorem on one of the sets U_i reduces to the change of variable theorem. Counting the points of $\gamma_{fol}^{-1}(L),\ L \in \mathcal{A}(2,1), L \in \gamma_{fol}(\bigcup U_i)$, we recognize $|\mu|(\mathcal{F}, L)$. Summing $\int_{U_i} |k|$ we get $\int_W |k|$ and, as the difference between W and $\bigcup U_i$ is the union of a set where $k = 0$ and a measure zero set, we get the statement of Theorem 2.1.

2.5. **Integral geometry of foliations of the Euclidean plane or of a flat torus.** We now give some applications of the foliated exchange theorem in planar domains. We let $|k|(m)$ denote the absolute value of the curvature of the leaf L_m of $\mathcal{F}$ through m.

Theorem 2.2. [15] *Let $D \in \mathbb{R}^2$ be the unit disc and $\mathcal{F}$ be an orientable foliation with isolated singularities, tangent to ∂D. Then*

$$K(\mathcal{F}) = \int_D |k| \geq 2\pi - 4,$$

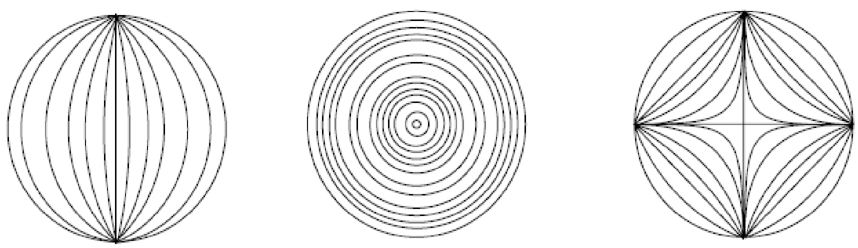
and the minimal value is achieved by the foliation (a) of Figure 9.

Proof. Let us choose an orientation of $\mathcal{F}$; this induces an orientation of $\partial D \setminus \operatorname{sing}(\mathcal{F})$. Among the singularities of $\mathcal{F}$ on ∂D, let A be the set of those where the orientation of ∂D changes. The set $A = a_1, a_2, \ldots, a_{2n}$ is finite and has an even number of points. Let G_e be the set of lines which meet D, do not meet A, and split A in two subsets containing an even number of points. Let G_o be the similar set of lines splitting A in two subsets of odd cardinality.

Recall the formula of Cauchy and Crofton (see [23])

Theorem 2.3. *Let C be a smooth plane curve, then*

$$2 \cdot \textit{length of } C \;=\; \int_{\mathcal{A}(2,1)} \#(C \cap L) dL$$



(a) $K(\mathcal{F}) = 2\pi - 4 \simeq 2.28$, (b) $K(\mathcal{F}) = 2\pi \simeq 6,28$ (c) $K(\mathcal{F}) = 2\pi + 8 - \sqrt{2} \simeq 2.97$

Figure 9. 3 examples of foliations of the disc

The formula of Cauchy and Crofton implies that the sum of the measures of G_e and G_o is 2π (the length of ∂D). If a line L is in G_e, then, if it contains no singularity of $\mathcal{F}$, we have $|\mu|(\mathcal{F}, L) \geq 1$ (see Figure 10).

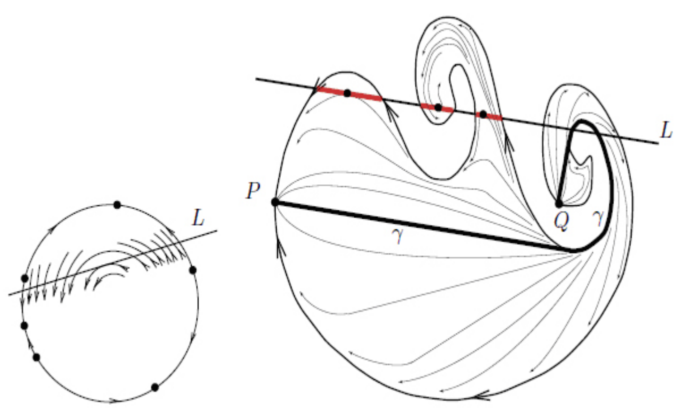


Figure 10. Forced contact and number of contact points of a line L and the foliation

Using the exchange theorem, we get the inequality

$$\int_D |k| \geq \text{measure}(G_e) = 2\pi - \text{measure}(G_o).$$

In order to finish the proof we need a lemma:

Lemma 2.1. *For any finite even subset A of the unit circle ∂D the measure of the set G_o of lines cutting A in two odd subsets satisfies*

$$\text{measure}(G_o) \leq 4.$$

$\square$

Remark 2.3. *When $A = \{a, -a\}$ is made of two opposite points,* measure$(G_o) = 4$. *When $A = \emptyset$, then* measure$(G_o) = 0$. *When A is the union of the vertices of a regular $2n$-gon, then* measure(G_o) *goes to π when n goes to infinity.*

The proof of the lemma is elementary but technical and can be found in [15].

Now let $D \subset \mathbb{R}^2$ be a domain homeomorphic to a disc and with a piecewise $\mathcal{C}^2$ boundary ∂D.

Definition 2.2. *The internal distance $d(m_1, m_2)$ of two points m_1 and m_2 is $d(m_1, m_2) =$*

$$\inf\{\mathit{length}(\gamma) | \gamma : [a, b] \to D \text{ a regular curve}, \gamma(a) = m_1,\ \gamma(b) = m_2\}$$

where length(γ) is the length of the curve γ.

In this way, we get a metric on D. In fact the assumptions made on D imply that given the two end points, there exists exactly one minimizing curve joining them. Such a curve will be called a *geodesic of D*.

Definition 2.3. *The diameter of D is defined by*

$$d = \sup\{d(m_1, m_2) | m_1 \in D, m_2 \in D\}.$$

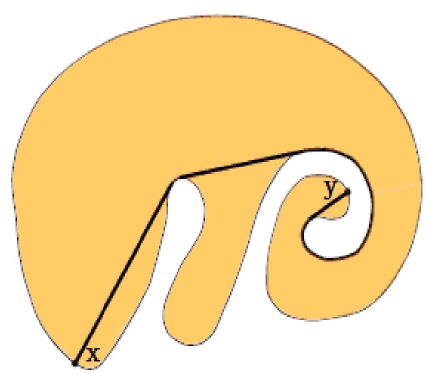


Figure 11. Diameter of a topological disc

Theorem 2.4. [16] *Let $\mathcal{F}$ be a foliation (by curves) of D, tangent to ∂D, with isolated singularities of positive index, not necessarily orientable. Then*

$$\int_D |k| \geq \text{length}(\partial D) - 2d.$$

Definition 2.4. *The index of an isolated singularity m of a non-orientable foliation of the plane is a half integer $\iota(m) \in \frac{1}{2}\mathbb{Z}$, which is half of the degree of the map*

$$\Phi_\varepsilon : S_\varepsilon(m) \to \mathbb{P}^1$$

associating with a point q of a small enough circle centered at m the direction of the line $T_q\mathcal{F}$ (if the singularity is orientable, the index is the usual one).

Notice that a Poincaré-Hopf theorem for non-orientable foliations of the disc is valid (see the appendix of [13]).

Proof. First, let us show that we can eliminate the case when $\mathcal{F}$ has a singularity of index one, studying only the case where $\mathcal{F}$ has two singularities of index $\frac{1}{2}$ which are of sunset type (see Figure 12, on the right).

All singularities of positive index can be substituted by a source/sink or a sunset singularity without increasing the total curvature of the foliation by more than a given ε. This can be done by considering on the boundary of a small disc D_r of radius r an homotopy between the "angle" function determined by $\mathcal{F}$ and the "angle" function of one of the models of Figure 12.

A source/sink can be replaced by two sunsets using the modification indicated in Figure 12. A center can be replaced by a sink (see Figure 13).

A singularity of positive index with m petals can be split first in m center-singularities and a saddle with m separatrices (see Figure 14).

A four-separatrices saddle can be replaced by two 3-separatrices saddles by a Whitehead transformation (see Figure 15). The same construction allows us to replace a saddle with n separatrices by one with $(n-1)$ separatrices and one with 3 separatrices.

By induction we can, by a perturbation in a neighborhood of the singularities which do not perturb the total curvature very much, eliminate all the previously met singularities which are not sunsets or saddles. We can now use a perturbation in a small annulus contained in a neighborhood of a singular point to get in restriction to the inner circle a line field turning with constant speed. We can now extend the latter foliation to the concentric circles of a disc neighborhood of the singularity using homotheties. This construction provides a foliation of the previous list.

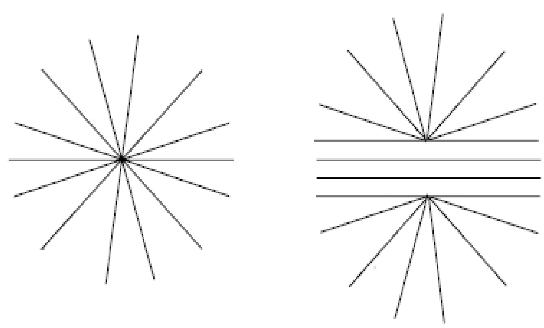


Figure 12. Transformation of a source/sink into two sunsets

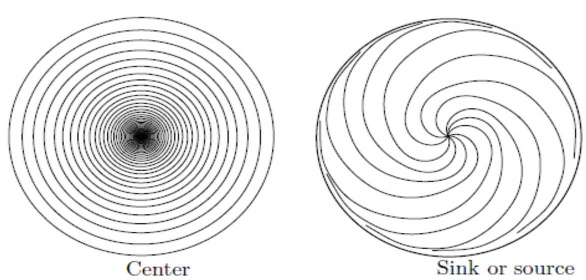


Figure 13. Center and sink or source

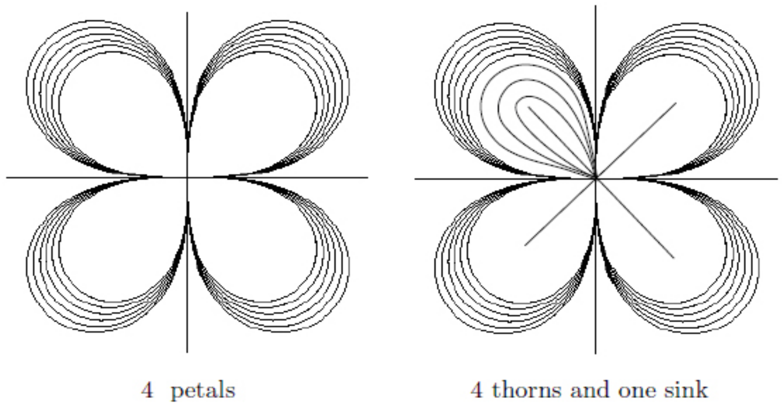


Figure 14. A 4-petals singularity and its modification

Eventually we obtain a foliation the singularities of which are only sunsets and 3-separatrices saddles.

Figure 15. A Whitehead transformation

Let P and Q be two sunsets of $\mathcal{F}$, and γ be a geodesic of D joining P to Q. We need to estimate the number of contact points of $\mathcal{F}$ with an affine line L. All lines, except a set of measure zero, meet the disc D in a finite number of segments.

Let $[a,b]$ be a connected component of $L \cap D$ such that $[a,b] \cap \gamma = \emptyset$. Then $[a,b]$ divides D into two discs, one of them containing P and Q. In the other disc, $\mathcal{F}$ is orientable, and therefore there is at least one point of contact between $\mathcal{F}$ and the segment $[a,b]$ (see Figure 10).

Let $n(L)$ be the number of segments of $L \cap D$ in which L meets γ, and $c(L)$ the number of segments of $D \cap L$ which do not. Then we have

$$|\mu|(\mathcal{F}, L) \geq c(L).$$

Cauchy-Crofton formula yields

$$\int_{\mathcal{A}(2,1)} \#\{\text{components of } L \cap D\} = \frac{1}{2} \int_{\mathcal{A}(2,1)} \#\{L \cap \partial D\} = \text{length}(\partial D).$$

Applying the Cauchy-Crofton formula to the arc γ, we get length $(\gamma) = \frac{1}{2} \int_{\mathcal{A}(2,1)} \#(L \cap \gamma)$. Then we obtain

$$\text{length}(\partial D) = \int_{\mathcal{A}(2,1)} n(L) + c(L) = \int_{\mathcal{A}(2,1)} \#(L \cap \gamma) + \int_{\mathcal{A}(2,1)} c(L).$$

Using the exchange theorem and the inequality on $|\mu|(\mathcal{F}, L)$ we get

$$\text{length}(\partial D) \leq 2 \cdot \text{length}(\gamma) + \int_{\mathcal{A}(2,1)} |\mu|(\mathcal{F}, L) = 2 \cdot \text{length}(\gamma) + \int_D |k|.$$

□

With the same techniques, one can obtain inequalities for foliations of a compact flat annulus, and for foliations of a disc extending a given line field defined on the boundary. In the second case, a sort of "length" of the envelope of the one-parameter family of affine lines defined by the boundary condition will play a role (see [16]).

2.6. **Tight foliations in dimension 2.** When a foliation achieves equality in the inequality of Theorem (2.4), we call it *tight*.

When the disc D is not convex, we can show there does not exist tight foliations tangent to ∂D with singularities of positive index. This comes from the fact that if $P \in \partial D$ is a point of inflexion, and a regular point of $\mathcal{F}$, then there is an open set of affine lines that have more than one contact point with $\mathcal{F}$ in a neighborhood of P. But we can exhibit a sequence $\mathcal{F}_n$ of foliations of D satisfying the hypothesis of our theorem such that

$$\lim_{n\to\infty} \int_D |k| = \text{length}(\partial D) - 2d.$$

We can think of the limit of this sequence of foliations as a foliation all leaves of which have corners along ∂D, in order to force on ∂D all the critical points of the orthogonal projections of the leaves on lines; see Figure 16.

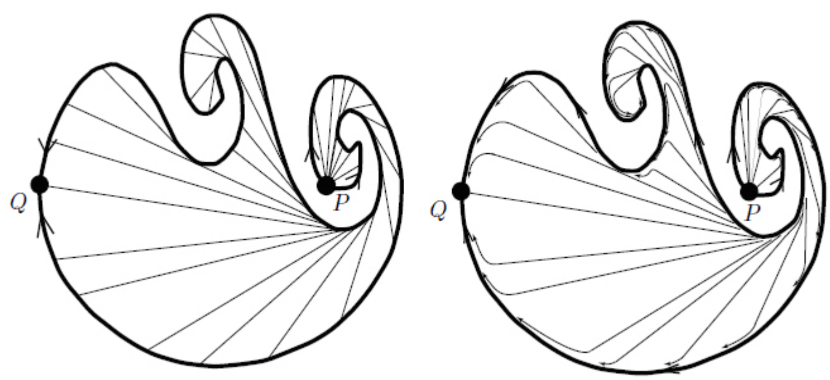


Figure 16. A tight singular foliation $\mathcal{F}$; a non-singular foliation $\mathcal{F}_n$ close to $\mathcal{F}$

We have seen that the foliated exchange theorem and some topological analysis of the foliation provide inequalities. Do there exist foliations achieving the equality case? We have called such foliations "tight". An example of a positive result is the following:

Theorem 2.5. *Let A be a plane annulus limited by two convex curves C_1 of length δ_1 and C_2 of length δ_2. We suppose that C_2 is the "inner" one (the Cauchy-Crofton formula then implies that $\delta_1 > \delta_2$). Then the leaves of the tight foliation of the annulus (tangent to the boundary) are either closed convex curves isotopic in A to C_1 (and C_2) or locally convex curves spiraling towards convex curves isotopic to C_1 (see the figure below). The total curvature of the foliation is, in that case*

$$\int_A |k| = \delta_1 - \delta_2.$$

Proof. Using the Cauchy-Crofton formula, we know that the set B of affine lines intersecting C_1 and not intersecting C_2 has measure $\delta_1 - \delta_2$. Such a line L intersects the annulus in a segment I. The foliation $\mathcal{F}$ is not transverse to the interior of I, otherwise the boundary of C_1 and I would form a Whitehead disc for $\mathcal{F}$, which is impossible as $\mathcal{F}$ has no singularity. Then

$$|\mu|(\mathcal{F}, L) \geq 1,$$

so the total curvature of $\mathcal{F}$ is greater than or equal to the measure of B. The equality is achieved for the foliations described in the theorem, as they satisfy

$$L \in B \Rightarrow |\mu|(\mathcal{F}, L) = 1;$$

$$L \notin B \Rightarrow |\mu|(\mathcal{F}, L) = 0.$$

□

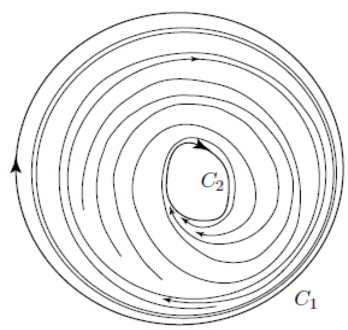


Figure 17. Tight foliation of a plane annulus with convex boundary curves

In [14], the reader will find a study of tight (in their isotopy class) foliations of the torus T^2.

2.7. **Foliations by geodesics or by curves of constant curvature.** Clearly the only foliation by geodesics of $\mathbb{R}^2$ are families of parallel line. In a domain of $\mathbb{R}^2$, a disc for example, there are many more.

In a torus $\mathrm{T}^2 = \mathbb{R}^2/\Lambda,\ \Lambda = \{u_1\mathbb{Z} \oplus u_2\mathbb{Z}\} \subset \mathbb{R}^2$, the only totally geodesic foliations are quotient of foliations of $\mathbb{R}^2$ by parallel lines. Leaves maybe all closed or all dense depending on the slope of the line (using the basis u_1, u_2 of $\mathbb{R}^2$). All the leaves of a foliation of T^2 cannot be circles, as then the foliation should have singularities of positive index.

Exercise. Prove that even if we accept isolated singularities, it is impossible to construct a foliation of a torus $\mathrm{T}^2 = \mathbb{R}^2/\Lambda$ by circles.

A sphere $\mathbb{S}^2$ cannot admit a foliation with geodesic leaves.

Exercise. Find two proofs of this statement.

Nevertheless, a pencil of geodesic circles provides a geodesic foliation of the sphere with two singular points.

Exercise. Prove that there exists no totally geodesic foliation of a compact hyperbolic surface (of constant curvature -1). Using the fact a singular foliation, with a finite number of singular points, of an hyperbolic surface should admit singularities of negative index, explain why singular (with a finite number of singular points) geodesic foliations of hyperbolic surface do not exist. To simplify the proofs, the reader may suppose that the foliations are orientable.

3. Codimension one foliations of manifolds of dimension 3 of constant curvature

For sake of simplicity, we suppose that the ambient 3-manifold is oriented. We suppose that the foliation is also oriented (the leaves have compatible orientations); it is therefore also transversely oriented.

Examples:

- parallel planes in $\mathbb{R}^3$ and the quotient foliations of $\mathrm{T}^3 = \mathbb{R}^3/\Lambda$. In the later case, leaves can be dense planes, dense cylinders, (closed) tori.
- Pencils of spheres in $\mathbb{S}^3$.
- Reeb foliation of $\mathbb{S}^3$ (see [22]). This is a very important example, as Novikov ([21]) proved that any foliation of $\mathbb{S}^3$ should contain a Reeb component (see [6] for an introduction to this difficult theorem).

The first object we need is the model Reeb foliation of the solid torus $D^2 \times S^1$. To obtain it we will construct a foliation of $D^2 \times \mathbb{R}$ invariant under unit translations in $\mathbb{R}$ (we can visualize $D^2 \times \mathbb{R}$ as a vertical solid cylinder). In the vertical band $[-1, 1] \times \mathbb{R}$ of the (x, z)-plane, consider a convex curve asymptotic to both sides of the band. By revolution around the z-axis we obtain a convex surface asymptotic to the boundary of the cylinder (on the $z \to +\infty$ side). Translating it vertically, we foliate the solid cylinder. By construction the foliation is invariant under vertical translation and then gives a foliation of the solid torus $T = (D^2 \times \mathbb{R}/(2\pi \cdot \mathbb{Z})$; see Figure 18.

We need now to recall that the sphere $\mathbb{S}^3$ can be obtained as the union of two solid tori intersecting along their common boundaries.

For that, visualize $\mathbb{S}^3$ as the unit sphere of $\mathbb{C}^2$. It is defined by the equation $|z_1|^2 + |z_2|^2 = 1$ it contains the torus of equations $|z_1|^2 = \frac{1}{2}; |z_2|^2 = \frac{1}{2}$. The solid tori T_1 and T_2 are given by the inequalities $\mathrm{T}_1 = \{|z_1|^2 \geq \frac{1}{2}, |z_2|^2 = 1 - |z_1|^2\}$ and $\mathrm{T}_2 = \{|z_2|^2 \geq \frac{1}{2}, |z_1|^2 = 1 - |z_2|^2\}$.

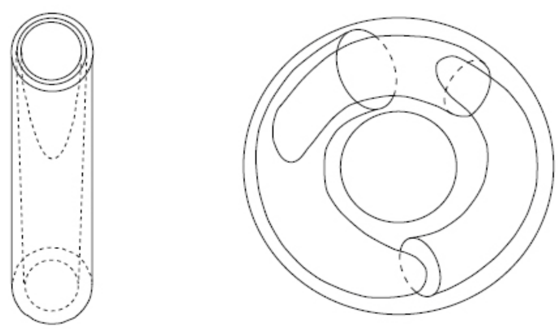


Figure 18. Reeb component

The core of the two tori are respectively the circle C_1 of equation $z_1 = 0$, and the circle C_2 of equation $z_2 = 0$. We now want to plug a Reeb component in each solid torus. Using again the fact that the unit sphere is embedded in $\mathbb{C}^2$, we can do that directly in $\mathbb{S}^3$.

We demand that the traces of the foliation are Poincaré component on the annuli of the spheres of the pencils of base circles respectively C_1 and C_2 obtained by taking away respectively the trace of the solid torus T_2 or the trace of the solid torus T_1. We may even chose one Poincar component on one anulus of one sphere of the pencil of base circle C_1 and rotate it by the maps $(z_1, z_2) \mapsto (z_1, e^{i\theta} z_2)$, and similarily rotate one Poincar component on a sphere of the second pencil by maps $(z_1, z_2) \mapsto (e^{i\theta} z_1, e^{i\theta} z_2)$.

3.1. **General results in dimension 3.** - Integrability: Frobenius' theorem.

To a foliation of a 3-dimensional manifold is associated a plane field on the manifold: $\mathcal{P} = \{P(m) = T_m(L(m)\}$ where $L(m)$ is the leaf of the foliation $\mathcal{F}$ which goes through the point m. It is not true that to every plane field $\mathcal{P}$ is associated a foliation $\mathcal{F}$ such that $\mathcal{P} = \{P(m) = T_m(L(m)\}$ where $L(m)$. The plane field should satisfy an *integrability condition* called the Frobenius condition. One way to express it is using a differentiable 1-form ω such that at each point $ker(\omega(m)) = P(m)$. Then it writes

$$\omega \wedge d\omega = 0. \tag{1}$$

Example 3.1. *A non-integrable plane field*

Let us consider on $\mathbb{R}^3, ($*basis* $(e_1, e_2, e_3)\,)$ *an horizontal vector field which twists at the same speed it goes up*

$$X(x, y, z) = (e^{iz}, 0),$$

where $(e^{iz}, 0) = (\cos z, \sin z, 0)$. *The form* ω *satisfies* $\omega(X) \equiv 0$, $\omega(m)(v) = 0\, \forall m \forall v \in P(m)$. *Therefore* $\omega(x, y, z) = \cos z dx + \sin z dy$. *We can first check that* $d\omega = \sin z dx \wedge dz - \cos z dy \wedge dz$ *and therefore that* $\omega \wedge d\omega = -cos^2 z dx \wedge dy \wedge dz - \sin^2 z dx \wedge dy \wedge dz = -dx \wedge dy \wedge dz$.

One geometrical way to see that the plane field is non-integrable is to check that the horizontal lines twisting at speed z along any vertical lines are tangent to the plane field $\mathcal{P}$. The surface they generate, an helicoid with axis the vertical line should therefore be a leaf. But two such helicoids would intersect, which is impossible for leaves of a foliation.

Another way to see that the plane field $\mathcal{P}$ cannot be integrable is using the Gauss map γ of the plane field, that is the map which, to $(m, P(m))$ associates the (a choice of orientation is necessary) vector $N(m)$ normal to $P(m)$ at m. We need to compute the derivative of this Gauss map in the direction of $P(m)$. In view the symmetries of the construction it is enough to perform the computation at the origin $(0, 0, 0, 0)$. Of course we chose $N(m) = X(m)$. We get, using e_2, e_3 as basis of $P((0, 0, 0))$

$$d\gamma = \begin{pmatrix} 0 & -\sin z \\ 0 & \cos z \end{pmatrix}.$$

This matrix is not symmetric as it should be if the plane field would have been integrable. Recall that performing the same computation of the plane tangent to the leaf of a foliation would provide the derivative of the Gauss map of the leaf of the foliation through the point which has a symmetric matrix when using an orthonormal basis of the tangent space to the leaf at the point.

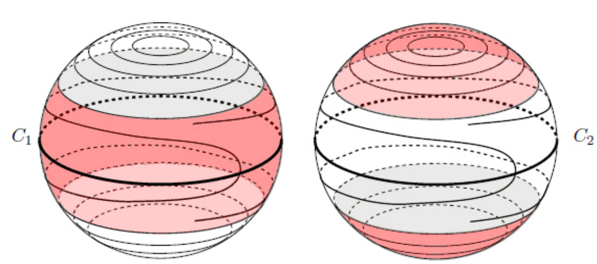


Figure 19. Sections of the Reeb foliation by a sphere of the pencil $\mathcal{P}_1$ and by a sphere of the pencil $\mathcal{P}_2$. The traces of the solid torus T_1 on the spheres are shadowed

3.2. **Integral geometry of codimension 1 foliations of space-forms of dimension 3.** The leaves are now surfaces. Let us start with the simplest case: $\mathcal{F}$ is a foliation of a domain $W \subset \mathbb{R}^3$. Let L be a leaf of $\mathcal{F}$. At each point m of the oriented surface L is associated a unit vector: $N(m)$. This defines a map $\gamma : L \to \mathbb{S}^2$. The differential $d\gamma(m)$ maps T_mL to itself. Using an orthonormal basis of T_mL, its matrix is symetrical. The eigenvalues of $d\gamma$ are called the principal curvatures of L at m. We note them $k_1(m)$ and $k_2(m)$. The *Gauss curvature* at m is $K(m) = k_1(m) \cdot k_2(m)$ and the mean curvature of L at m is $H(m) = (k_1 + k_2)/2$. The eigen-directions of $d\gamma$ are called principal directions. When $k_1(m) = k_2(m)$, the point m is an *umbilic* (at an umbilic, all directions are principal). As, through any point of W goes a leaf of $\mathcal{F}$, we can now consider $K(m)$, the Gauss curvature at m of the leaf through m, and $H(M)$ the mean curvature at m of the leaf through m, as functions on W.

3.2.1. *The easiest example:* T^3. To a foliation $\mathcal{F}$ we can associate a one-parameter family of maps $\Phi_t : \mathrm{T}^3 \to \mathrm{T}^3$:

$$\Phi_t(m) = m + t \cdot N(m),$$

where N(m) is the normal vector at m to the leaf of $\mathcal{F}$ which passes through m.

Let us compute, using an orthonormal basis of $T_m\mathrm{T}^3$ with the first two vector contained in principal directions, the jacobian determinant $det(d\Phi_t(m))$:

$$det \begin{pmatrix} 1 + tk_1 & 0 & 0 \\ 0 & 1 + tk_2 & 0 \\ * & * & 1 \end{pmatrix} = 1 + t(k_1 + k_2) + t^2 K.$$

The integral of this jacobian determinant, $vol\mathrm{T}^3 + t \int_{\mathrm{T}^3}(k_1 + k_2) + t^2 \int_{\mathrm{T}^3} K$ is the volume of T^3. Moreover, when t is small enough, Φ_t is a diffeomorphism.

We get therefore Asimov's theorem [1]:

Theorem 3.1. *Let $\mathcal{F}$ be an orientable codimension 1 foliation of* T^3. *Then*

$$\int_{\mathrm{T}^3} H dv = 0 \quad and \quad \int_{\mathrm{T}^3} K dv = 0.$$

In [1] D. Asimov[1] then considered foliations of manifolds of constant sectional curvature; in dimension 3 these are quotients of $\mathbb{S}^3$ or $\mathbb{H}^3$.

We will prove this result when the 3-manifold is $\mathbb{S}^3$ using an idea of Asimov that Milnor (see [19]) used to prove that any vector field on $\mathbb{S}^2$ has to have a zero.

The unit vector $N(m_0)$ is now in $T_{m_0}\mathbb{S}^3$. Viewing $\mathbb{S}^3 = \mathbb{S}^3_1$ as the unit sphere of $\mathbb{R}^4$ we can also see $N(m_0)$ as a vector in $\mathbb{R}^4$. The map Φ_t now maps the unit sphere to the sphere $\mathbb{S}^3_{\sqrt{1+t^2}}$ of radius $\sqrt{1+t^2}$. Let us consider an auxiliary field N_0 defined in a neighbourhood of m: restricted to the totally geodesic sphere Σ tangent to L at m_0 it is constant and coincide with $N(m_0)$, then extend it by parallel transport on meridians orthogonal to Σ.

The jacobian matrix at m_0 of the map

$$\begin{aligned} &T_{m_0}\mathbb{S}^3_1 \to T_{m_0+tN_0}\mathbb{S}^3_{\sqrt{1+t^2}} \\ &m \mapsto m + N_0 \end{aligned}$$

[1] D. Asimov was the first to consider integrals of the Gauss curvature of the leaves of a codimension 1 foliation. He considered first foliations of compact 3-manifolds of constant curvature. Higher dimensional results where then independently proved in by Asimov [1] and, after reading a manuscript by Asimov dealing with the 3-dimensional case, by Brito Langevin and Rosenberg [2], [3].

is

$$\begin{pmatrix} 1 & 0 & O \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{1+t^2} \end{pmatrix}.$$

The jacobian matrix of the map at m_0 is, provided the source and target basis start with two unit vectors contained in the principal direction at m_0,

$$\begin{pmatrix} 1+tk_1 & 0 & O \\ 0 & 1+tk_2 & 0 \\ 0 & 0 & \sqrt{1+t^2} \end{pmatrix},$$

where k_1 and k_2 are the eigenvalues of the second fundamental form of $L \subset \mathbb{S}^3$. The determinant of this matrix is $= \sqrt{1+t^2} \cdot [1 + t(k_1 + k_2) + t^2 k_1 k_2$. The volume of $\mathbb{S}^3_{\sqrt{1+t^2}}$ is $\sqrt{1+t^2}]^3 \cdot vol(\mathbb{S}^3_1)$. Integrating on $\mathbb{S}^3_1$ the jacobian determinant of Φ_t and using the fact that, for t small enough, Φ_t is a diffeomorphism, we get:

$$\int_{\mathbb{S}^3} H dv = 0 \quad \text{and} \quad \int_{\mathbb{S}^3} K_e dv = vol(\mathbb{S}^3) = 2\pi^2,$$

where H is the mean extrinsic curvature of L, and K_e its extrinsic Gauss curvature.

Using Klein model of the hyperboloid in Lorentz space, the same proof works also in hyperbolic manifolds; in this case:

$$\int_M H dv = 0 \quad \text{and} \quad \int_M K_e dv = -vol(M).$$

Formulas when the constant curvature is c are:

Theorem 3.2. [1] *Let $\mathcal{F}$ be a codimension 1 foliation of a 3-manifold of constant curvature c, then*

$$\int_M H dv = 0 \quad \textit{and} \quad \int_M K_e dv = c \cdot vol(M),$$

where H is the mean extrinsic curvature of L, and K_e its extrinsic Gauss curvature.

3.2.2. *Total curvature.* Integral geometry provides new proofs of Asimov's theorems, and deals also with the *total curvature* of the foliation, that is the integral $\int_M |K| dv$.

3.2.3. *Contacts with affine planes and the exchange theorem.* We will repeat, adding one dimension, what we did for foliations of domains of $\mathbb{R}^2$.

Let H be an affine hyperplane of $\mathbb{R}^3$. The trace $\mathcal{F}|_H$ of $\mathcal{F}$ on H is generically a foliation of $W \cap H$ with only isolated singularities. We call this finite set of singular points $\Sigma(\mathcal{F}|_H)$.

In fact, generically these singularities are non-degenerate, and are of one of the two following types: center or saddle. We attribute signs to these singular points of the trace foliation $\mathcal{F}|_H$ on H:

$$\varepsilon(\text{saddle}) = -1 \textit{ and } \varepsilon(\text{center}) = +1.$$

(see Milnor's book [20] for a definition of the index of a singular point of a vector field). When the leaves of the foliation are locally the levels of a function with a non-degenerated critical point at m, of index $\iota(m)$ (see Milnor's book [18]), this sign is also the index of the gradient of f at m: $\text{index}_{\text{grad}(f)}(m) = (-1)^{\iota(m)}$.

Definition 3.1. *The number $|\mu|(\mathcal{F}, H)$ is the number of singular points of $\mathcal{F}|_H$.*

When $|\mu|(\mathcal{F},H)$ is finite, and the singularities are all non-degenerate, the number $\mu^+(\mathcal{F},H)$ is

$$\mu^+(\mathcal{F},H) = \sum_{m\in\,\Sigma(\mathcal{F}|_H)} \varepsilon(m).$$

Remark 3.1. *A singular point m of $\mathcal{F}|_H$ is a point where the leaf L_m is tangent to H. We can also locally project L_m on the normal in m to H (and to L_m). We get a function which is in general a Morse function (see Milnor's book* [18], *for which the Morse index of m satisfies*

$$(-1)^{\textit{Morse index of } m} = \varepsilon(m).$$

The sign $\varepsilon(m)$ is, when the dimension of the leaves of $\mathcal{F}$ is even, the sign of the Gauss curvature of L_m at m.

We will call the integral $\int_W |K|$ (or $\int_W |k|$ when W is of dimension 2) the *total curvature of* $\mathcal{F}$.

Theorem 3.3. *(Foliated exchange theorem).*

$$\int_W |K| = \int_{\mathcal{A}(3,2)} |\mu|(\mathcal{F},H).$$

Moreover, if one of the previous integrals is finite, then

$$\int_W K = \int_{\mathcal{A}(3,2)} \mu^+(\mathcal{F},H).$$

To prove this theorem, we will define, as in $\mathbb{R}^2$, the *polar curves* of the foliation and a foliated Gauss map.

Polar curves. Let us call p_ℓ the orthogonal projection on the line ℓ. The critical points of the the restriction of p_ℓ to a leaf L of $\mathcal{F}$ are isolated on the leaf L as soon as the curvature of the leaf in non-zero at that point.

Definition 3.2. *We will call polar set associated to the line direction ℓ the closure of the union of the critical points of p_ℓ restricted to all the leaves of the foliation*

$$\Gamma(\mathcal{F},\ell) = \overline{\cup_L \mathit{Crit}(p_\ell|_L)}.$$

Defining a map $\gamma W \to \mathbb{P}^1$ by $\gamma(m) =$ the line normal to $\mathcal{F}$ at m, we see that for ℓ out of a measure zero set of $\mathbb{P}^1$, the set $\Gamma(\mathcal{F},\ell)$ is a regular curve. Moreover, points of W where $K \neq 0$ are non-critical for γ, therefore, through such a point passes exactly one arc of a polar set.

Proposition 3.1. *Near points where $K \neq 0$ the polar set $\Gamma(\mathcal{F},\ell)$ is an arc transverse to $\ell^\perp = T_m\mathcal{F}$.*

Remark 3.2. *When $\Gamma(\mathcal{F},T_m\mathcal{F}^\perp)$ is tangent to $T_m\mathcal{F}$ the Gauss curvature of the leaf L_m is zero, as, in that case, the differential of the Gauss map of the leaf L_m restricted to $T_m\Gamma(\mathcal{F},T_m\mathcal{F}^\perp)$ is zero.*

To prove the foliated exchange theorem we need to introduce a foliated Gauss map with values in $\mathcal{A}(3,2)$:

Definition 3.3.

$$\gamma_{\mathcal{F}}(m) = \textit{the affine plane tangent at } m \textit{ to } \mathcal{F}.$$

Proof. To compute the Jacobian determinant of the foliated Gauss map $\gamma_{\mathcal{F}}$ at a point $m \in W$ we will use, when $\Gamma(\mathcal{F}, T_m\mathcal{F}^{\perp})$ is transverse to $T_m\mathcal{F}$ in the domain, the frame u_1, u_2, u_3, where u_1, u_2 is an orthogonal basis of $T_m\mathcal{F}$, and u_3 is the unit vector tangent at m to $\Gamma(\mathcal{F}, T_m\mathcal{F}^{\perp})$. In $\mathcal{A}(3,2)$ we use at $\gamma_{\mathcal{F}}(m)$ the frame v_1, v_2, v_3, where v_1, v_2 form an orthogonal basis of the horizontal space at $\gamma_{\mathcal{F}}(m)$ of the Riemannian fiber bundle $\mathcal{A}(3,2) \to \mathbb{P}^2$, and where v_3 is a unit vector tangent to the fiber of $\mathcal{A}(3,2) \to \mathbb{P}^2$. In these bases, the matrix of $d\gamma_{\mathcal{F}}$ is

$$\begin{pmatrix} d\gamma_{\mathcal{F}}|_{L_m} & 0 \\ * & |\cos\phi| \end{pmatrix},$$

where ϕ is the angle between $T_m\Gamma_{\mathcal{F}}$ and $T_m\mathcal{F}^{\perp}$.

As the volume of the parallelogram determined by the frame u_1, u_2, u_3 is also $|\cos\phi|$, and as the map $d\gamma_{\mathcal{F}}|_{L_m}$ is just the Gauss-Kronecker map of the leaf L_m, the Jacobian determinant we are looking for is $|K|$.

On the one hand, when $\Gamma(\mathcal{F}, T_m\mathcal{F}^{\perp})$ is tangent to $T_m\mathcal{F}$, the Gauss-Kronecker curvature K is zero. On the other hand, using a frame split between $T_m\mathcal{F}$ and $T_m\mathcal{F}^{\perp}$, we see that at such a point the matrix of $d\gamma_{\mathcal{F}}$ is

$$\begin{pmatrix} d\gamma(m) & * \\ 0 & 1 \end{pmatrix},$$

where in the formula $d\gamma$ is the Gauss map of the leaf L_m. As the rank of $d\gamma(m)$ is 1, the point m is critical for $\gamma_{\mathcal{F}}$; by Sard's theorem the measure of the images by $\gamma_{\mathcal{F}}$ of these points is zero. □

3.3. **Integral geometry for codimension one foliations of spaces of constant curvature in dimension 3.** In dimension 3, when the foliated space is a domain W contained in $\mathbb{S}^3$ or $\mathbb{H}^3$, one can also prove an exchange theorem, replacing the Gauss-Kronecker curvature by the determinant of the second fundamental form (that we will denote by K_e, the extrinsic Gauss-Kronecker curvature) obtained using the normal vector given by the orientation (in an orthonormal basis). The Gauss map γ has now its values in $\mathbb{P}^2$, or $\mathbb{S}^2$ if you prefer to suppose that the foliation is oriented. The foliated Gauss map has its values in the set of affine planes of $\mathbb{R}^3$: the affine Grassmann manifold $\mathcal{A}(3,2)$.

To replace the foliated Gauss map $W \to \mathcal{A}(3,2)$ when W is contained in $\mathbb{S}^3$ or $\mathbb{H}^3$, we need to replace the Euclidean affine planes by totally geodesic spheres $\Sigma \subset \mathbb{S}^3$ or by totally geodesic hyperbolic planes $H \subset \mathbb{H}^3$. The form of the theorem is the same for $W \subset \mathbb{H}^3$, $W \subset \mathbb{R}^3$, and $W \subset \mathbb{S}^3$. In each case the set $\mathcal{A}$ of totally geodesic subspaces of dimension 2 admits a measure invariant under the action of the isometries of the space [23, pp. 28, 307].

Theorem 3.4. *The total curvature of the codimension* 1 *foliation of* $W \subset \mathbb{R}^3$, $\mathbb{S}^3$ *or* $\mathbb{H}^3$ *is given by*

$$\int_W |K| = \int_{\mathcal{A}} |\mu|(\mathcal{F}, H),$$

where $\mathcal{A}$ is the set of affine planes, hyperbolic planes, or geodesic 2*-dimensional spheres.*

Proof. When W is contained in $\mathbb{R}^3$, the proof is exactly the same as in the plane (Theorem 2.1) as again the set $K = 0$ contains the critical points of both γ and $\gamma_{\mathcal{F}}$. In dimension 3, we will only accept a finite union of smooth curves as singular set of the foliation.

When W is contained in $\mathbb{S}^3$ or $\mathbb{H}^3$, we need to replace the orthogonal projections on lines used to prove the exchange theorem for surfaces in $\mathbb{R}^3$. A geodesic L defines a one-parameter family, called a *pencil* $\mathcal{P}_L$ of totally geodesic hypersurfaces: those orthogonal to it. In $\mathbb{H}^3$ a pencil is a foliation and defines a projection on the geodesic L. In $\mathbb{S}^3$ a pencil defines a foliation of $\mathbb{S}^3 \setminus \mathbb{S}^1$ and a projection of $\mathbb{S}^3 \setminus \mathbb{S}^1$ on $\mathbb{P}^1$.

Definition 3.4. *The polar curve $\Gamma_{\mathcal{P}}$ is the closure of the set of points where a hypersurface of the pencil $\mathcal{P}$ is tangent to the foliation.*

Remark 3.3. *As in the Euclidean case, $\Gamma_{\mathcal{P}}$ is, for almost all $\mathcal{P}$, almost everywhere a smooth curve.*

Definition 3.5. *The foliated Gauss map $\gamma_{\mathcal{F}} : W \to \mathcal{A}$ associates with a point $m \in W$ the totally geodesic hypersurface tangent at m to the leaf L_m of $\mathcal{F}$ through m.*

The computation of the Jacobian determinant of $\gamma_{\mathcal{F}}$ is the same as in the Euclidean case, observing that the totally geodesic surfaces orthogonal to the geodesic $L(m)$ through m orthogonal to L_m, and the totally geodesic surfaces through m, form two submanifold of $\mathcal{A}$ orthogonal in $\mathcal{A}$ for the natural Riemannian metric of $\mathcal{A}$. □

The following theorem is now a consequence of the fact that the intersection of a foliation of $\mathbb{S}^3$ with a generic totally geodesic $\mathbb{S}^2$ has at least two singular points.

Theorem 3.5. *Let $\mathcal{F}$ be a foliation of $\mathbb{S}^3$ having a finite number of singularities. Then*

$$\int_{\mathbb{S}^3} |K_e| \geq 2\pi^2.$$

Using the Poincaré-Hopf theorem on all the generic totally geodesic $\mathbb{S}^2$'s, we also prove the following theorem:

Theorem 3.6. *If one of the previous integrals is finite,then*

$$\int_{\mathbb{S}^3} K_e = 2\pi^2.$$

Remark 3.4. Mean curvature. *The mean curvature H of the leaves of a foliation of a compact 3-manifold M satisfy, whatever the metric of M is, $\int_M H dv = 0$.*

This can be proved computing the differential $d\theta$ of the area form θ of the leaves of the foliation. Using an othonormal moving frame e_1, e_2, e_3, where the first two unit vectors are contained in the principal direction of the second fundamental form of the leaves, and forms θ_i such that $\theta_i(e_j) = \delta_{ij}$ we see that $\theta = \theta_1 \wedge \theta_2$, and $d\theta = (k_1 + k_2)\omega$, where ω is the volume form of the ambient manifold M. The result is a consequence of the fact that the integral of an exact form is zero.

3.3.1. *Tight foliations.* A foliation of a riemannian manifold M will be called *tight* if its total curvature $\int_M |K_e|$ achieves the lower bound $inf_{\mathcal{F}} \int_M |K_e|$.

Not all Riemannian manifolds admit tight foliation. For example:

Theorem 3.7. *There does not exist any tight foliation of the sphere $\mathbb{S}^3$.*

Proof. We have seen before that the total curvature of a foliation $\mathcal{F}$ of $\mathbb{S}^3$ satisfies $\int_{\mathbb{S}^3} |K_e| \geq 2\pi^2$, because for a generic totally geodesic sphere $\Sigma \subset \mathbb{S}^3$ one has $|\mu|(\mathcal{F}, \Sigma) \geq 2$. We have also seen that

$$\int_{\mathbb{S}^3} K_e = 2\pi^2.$$

If a foliation $\mathcal{F}$ of $\mathbb{S}^3$ satisfies $\int_{\mathbb{S}^3} |K_e| = \int_{\mathbb{S}^3} K_e$, then the curvature function should satisfy $K_e \geq 0$. In $\mathbb{S}^3$ the intrinsic curvature K_i of an embedded surface satisfy $K_i = K_e + 1$ (one can perform the computation using the exponential map (see [24]).

Novikov's theorem states that the foliation has a Reeb component [6] with boundary a torus leaf L. The Gauss-Bonnet theorem applied to L states that $\int_L K_i = 0$. Then $\int_L K_e = -vol(L) < 0$, so the leaf has a point of negative (extrinsic) curvature K_e, contradicting the hypothesis.

The theorem will then be proved if we can show that

$$\inf \int_{\mathbb{S}^3} |K_e| = 2\pi^2.$$

Let us consider the singular foliation $\mathcal{P}$ of $\mathbb{S}^3$ defined by a pencil of geodesic 2-spheres. It has a one dimensional singular locus: a geodesic circle C. The trace of $\mathcal{P}$ on a geodesic sphere Σ transverse to C is a foliation with two singular points of index 1 (of type sink/source).

We will now shadow the foliation $\mathcal{P}$ by nonsingular ones, introducing a very thin Reeb component in a tubular neighborhood of C.

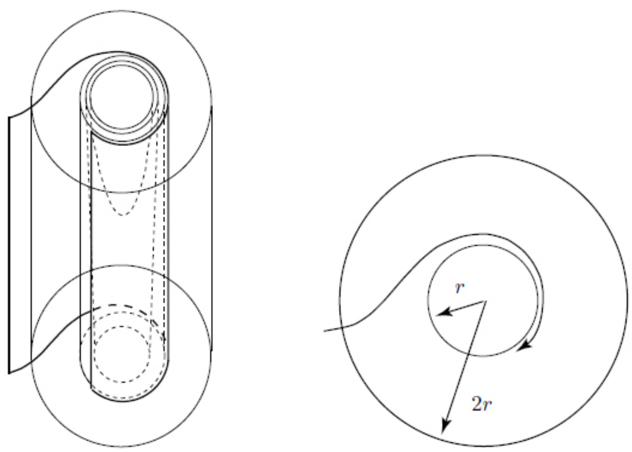


Figure 20. A piece of a thin Reeb component (in a solid cylinder R_1) and how the other leaves wrap around it in the region $R_2 \setminus R_1$; horizontal section of the foliation.

To construct the foliation in a tubular neighborhood $\mathrm{Tub}_{2r}(C)$ of radius $2r$ of C, we will first construct a model in the cylinder $D^2_{2r} \times \mathbb{R}$, invariant under vertical translations.

In the cylinder $D^2_r \times \mathbb{R}$ just put a Reeb component defined as above. In the annulus $D^2_{2r} \setminus D^2_r$, seen as a subset of the (x, y)-plane, consider a curve entering, normally to the boundary, into D^2_{2r} and spiraling towards the circle ∂D^2_r (see Figure 20).

The product of this curve by the vertical line is a surface of $\mathbb{R}^3$ entering normally the cylinder $D^2_r \times \mathbb{R}$ and spiraling toward the inner cylinder $D^2_r \times \mathbb{R}$. By rotation around the z-axis, we foliate the set $(D^2_{2r} \setminus D^2_r) \times \mathbb{R}$. So we get the desired foliation of the solid cylinder $D^2_r \times \mathbb{R}$.

The quotient by the vertical translations by vectors of length 2π is a foliation of $D^2_{2r} \times S^1$. Let us now map $D^2_{2r} \times S^1$ to the tubular neighborhood of (geodesic) radius 2r of C, mapping S^1 on C isometrically, and using the exponential map to map the discs D^2_{2r} centered on points $(0, 0, z) \in S^1$ onto totally geodesic discs normal to C. We obtain a foliation $\mathcal{F}_r$ which fits with $\mathcal{P}|_{\mathbb{S}^3 \setminus \mathrm{Tub}_{2r}(C)}$. The reader should now believe that

- The geodesic spheres Σ satisfy $|\mu|(\mathcal{F}_r, \Sigma) = 2$ if Σ intersects C with not too small an angle;
- There exists a uniform bound, independent of r, for the number $|\mu|(\mathcal{F}_r, \Sigma)$.

As the measure of the geodesic spheres that intersect C with an angle smaller than ε goes to zero with ε, we proved, using the foliated exchange theorem, that

$$\lim_{r \to 0} \int_{\mathbb{S}^3} |K_e| = 2\pi^2,$$

where $|K_e|$ is the absolute value of curvature function defined by the leaves of $\mathcal{F}_r$. $\square$

Exercise. Find an alternative proof of Theorem 3.7 using the singular foliation of $\mathbb{S}^3$ obtained filling the two Reeb components T_1 and T_2 of the beginning of Section 3 by spherical caps.

References

[1] Asimov, D. (1978), Average Gaussian curvature of leaves of foliations, Bulletin of the American Math. Soc., 84(1), pp.131-133.

[2] Brito, F., Langevin, R. and Rosenberg, H., (1977), Intégrales de courbure sur des variétés feuilletées, note aux C. R. Acad. Sc. Paris, 285, série A, pp.533-536.

[3] Brito, F., Langevin, R. and Rosenberg, H., (1981), Intégrales de courbure sur des variétés feuilletées, Journal of Differential Geometry, 16, pp.19-50.

[4] Candel, A. and Conlon, L., (2000)(2003), Foliations I, II, Amer.Math. Soc., Providence.

[5] Chevalley, C., (1946), Theory of Lie Groups, Princeton Univ. Press.

[6] Camacho, C. and Lins Neto, A., (1985), English version, translation S. Goodman: Geometric Theory of Foliations, Birkhaüser.

[7] do Carmo, M.,(1976), Differential Geometry of Curves and Surfaces, Prentice-Hall.

[8] Ehresmann, Ch.,(1944), Note Comptes Rendus Acad. Sci. Paris, 218, pp.955-957.

[9] Ehresmann, Ch., (1947), Note Comptes Rendus Acad. Sci. Paris, 224, pp.1611-1612.

[10] Godbillon, C., (1991), Feuilletages, Birkhaüser Basel.

[11] Kaplan, N., (1940), Regular curve families filling the plane I, Duke Math. Journal, 7, pp.154-185.

[12] Kneser, H., (1923), Reguläre Kurvenscharen auf die Ringflachen, Math. Annalen, 91, pp.135-154.

[13] Langevin, R., (1979), Integral geometry from Buffon to geometers of today, provisory version available at http://math.u-bourgogne.fr/topologie/langevin/index.html.

[14] Langevin, R., Feuilletages tendus, Bull. de la Soc. Math. France, 107, pp.271-281.

[15] Langevin, R. and Levitt, G., (1985), Sur la courbure totale des feuilletages des surfaces à bord, Bolletim da sociedade brasileira de matematica, 16(1), pp.1-13.

[16] Langevin, R. and Possani, C.,(1993), Total curvature of foliations, IIlinois Journal of Mathematics, 37(3), pp.508-524.

[17] Langevin, R. and Shifrin, T., (1982), Polar varieties and integral geometry, American Journal of Math., 104(3), pp.553-605.

[18] Milnor, J., (1963), Morse Theory, Annals of Mathematics Studies, 51, Princeton University Press.

[19] Milnor, J., (1978), Analytic proof of the hairy ball theorem and the Brouwer fixed point theorem, American Math. Monthly, 85(7), pp.521-524.

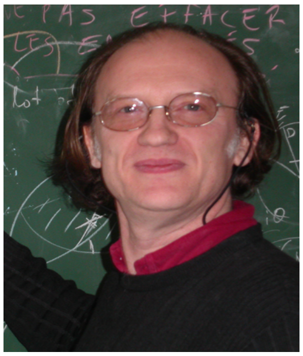
[20] Milnor, J., (1969), Topology from the Differential Viewpoint, The University Press of Virginia, second edition.

[21] Novikov, S.P., (1965), Topology of foliations, Trans. Moscow Math. Soc., pp.268-304.

[22] Reeb, G., (1952), Sur certaines propriétés topologiques des variétés feuilletées, in Publ. Inst. Math. Univ. Strasbourg 11, , Actualités Sci. Ind., 1183, Hermann & Cie., Paris, pp.5–89, 155–156.

[23] Santaló, L.A., (1976), Integral Geometry and Geometric Probability, Encyclopedia of mathematics and its applications. Addison Wesley.

[24] Spivak, M., (1979), A Comprehensive Introduction to Differential Geometry, Publish or Perish, Berkeley.



Rémi Langevin is a professor at the University of Burgundy (Dijon, France). He obtained his degrees from the University of Paris 11 (Orsay, France). He has been president of the French Mathematical Society. His main fields of interest are integral geometry, conformal geometry, dynamical systems, and foliations. He has published approximately 60 research articles and, with C. Bonatti, a book on dynamical systems (Astérisque, 1998). He has supervised 9 Ph.D. students. He particularly likes to convey mathematical ideas using pictures.